ASYMMETRIC TEMPERATURE FIELD OF AN UNBOUNDED CYLINDER WITH A MOVING HEATING LINE

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The temperature distribution in an unbounded hollow cylinder, a portion of whose inner surface is asymmetrically heated, is obtained under the assumption that the heating line moves at a certain speed toward the cylinder axis.

Problems involving the determination of the temperature field in axisymmetrically heated cylinders, where the heating line moves at a certain speed, have been examined by numerous investigators [1-4]. The stresses arising in the cylinder material under the action of the temperature were obtained likewise in [1, 3]. Not less important is the problem of determining the temperature field and the corresponding stress field in cylinders, only a portion of whose lateral surface is subjected to asymmetric heating, since it is heating of this type that is frequently observed in practice in the operation of power systems [5, 6].

In the present paper, the temperature distribution in a hollow infinite cylinder is obtained under the following assumptions: the cylinder is not heated in the initial state; at a moment of time t, a portion of the inner surface defined by the coordinates $z < 0$ and $\gamma < \varphi < (2\pi - \gamma)$ is heated to a temperature T_0, while the portion defined by the coordinates $(2\pi - \gamma) \le \varphi \le \gamma$ remains unheated; the heating line moves at a certain speed V in the positive direction of the z axis (Fig. 1).

Having expressed the temperature at the surface $r = a$ in the form of a Fourier series where the n-th coefficient is denoted by f_n, the solution of the heat equation:

$$\varkappa \Delta T = \frac{\partial T}{\partial t},$$

$$\Delta(\ ) = \frac{\partial^2(\ )}{\partial r^2} + \frac{1}{r}\frac{\partial(\ )}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\ )}{\partial \varphi^2} + \frac{\partial^2(\ )}{\partial z^2} \tag{1}$$

is sought in the form

$$T = T(r, \varphi, z, t) = \sum_{n=1}^{\infty} T_n(r, z, t)\cos n\varphi. \tag{2}$$

We abstain from examining the value $n = 0$, since it refers to an axisymmetric temperature distribution, which has been thoroughly analyzed in [1-3].

Let us introduce the dimensionless coordinates

$$\rho = \frac{r}{b}, \quad \zeta = \frac{z}{b}, \quad \tau = \frac{\varkappa t}{b^2}, \quad u = \frac{bv}{\varkappa}, \quad \beta = \frac{a}{b}.$$

For determining the coefficients of series (2), we obtain on the basis of equality (1) the following differential equation:

$$\Delta_n T_n = \frac{\partial T_n}{\partial t},$$

$$\Delta_n(\ ) = \frac{\partial^2(\ )}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial(\ )}{\partial \rho} - \frac{n^2(\ )}{\rho^2} + \frac{\partial^2(\ )}{\partial z^2} \tag{3}$$

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with the boundary and initial conditions

$$
\begin{aligned}
&T_n(\rho, \zeta, 0) = 0, \quad \beta \leqslant \rho \leqslant 1;\\
&T_n(\beta, \zeta, \tau) = f_n(\zeta - u\tau);\\
&\frac{\partial T_n}{\partial \rho} + hT_n = 0, \quad \rho = 1;\\
&f_n(\zeta - u\tau) = \begin{cases} f_n, & \zeta < u\tau, \\ 0, & \zeta > u\tau. \end{cases}
\end{aligned}
\tag{4}
$$

In order to obtain the form of function T_n, we apply to Eq. (3) the apparatus of integral transforms, namely: Fourier transforms with respect to the ζ coordinate, and Laplace transforms with respect to the τ coordinate. As a result, we obtain an equation and the corresponding boundary conditions for determining a function of only the ρ coordinate.

By applying to function T_n Fourier transforms with respect to ζ

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T_n \exp(ip\zeta)\, d\zeta = \overline{T}_n,$$

we obtain for Eq. (3):

$$\frac{\partial^2 \overline{T}_n}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \overline{T}_n}{\partial \rho} - \frac{n^2 \overline{T}_n}{\rho^2} - p^2 \overline{T}_n = \frac{\partial \overline{T}_n}{\partial t}. \tag{5}$$

For the transforms, the boundary and initial conditions (4) take the form

$$
\begin{aligned}
&\overline{T}_n(\rho, p, 0) = 0, \quad \beta \leqslant \rho \leqslant 1;\\
&\frac{\partial \overline{T}_n}{\partial \rho} + h\overline{T}_n = 0, \quad \rho = 1;
\end{aligned}
\tag{6}
$$

$$\overline{T}_n(\beta, p, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_n(\zeta - u\tau) \exp(ip\zeta)\, d\zeta = \sqrt{2\pi}\, f_n \exp(ipu\tau)\, \delta_-(p).$$

Here, as in [7], it is assumed that

$$\delta_-(x) = \frac{1}{2}\, \delta(x) + \frac{1}{2\pi i x}.$$

Now we apply Laplace transforms to Eq. (5). This equation then takes the form

$$\frac{d^2 T_n^*}{d\rho^2} + \frac{1}{\rho} \frac{dT_n^*}{d\rho} - \frac{n^2 T_n^*}{\rho^2} - (p^2 + \alpha)\, T_n^* = 0, \tag{7}$$

where

$$T_n^* = \int_0^{\infty} \overline{T}_n \exp(-\alpha\tau)\, d\tau.$$

The boundary conditions (6) reduce to the form:

$$\frac{dT_n^*}{d\rho} + hT_n^* = 0, \quad \rho = 1;$$

$$T_n^*(\beta, p, \alpha) = \sqrt{2\pi} \int_0^{\infty} \delta_-(p) f_n \exp(ipu - \alpha)\tau d\tau = \sqrt{2\pi}\, \delta_-(p) f_n \frac{1}{\alpha - ipu}, \quad \mathrm{Re}(\alpha - ipu) > 0. \tag{8}$$

Equation (7) is the Bessel equation. Its solution is

$$T_n^* = AI_n(\xi\rho) + BK_n(\xi\rho), \quad \xi = \sqrt{p^2 + \alpha}.$$

We determine the coefficients A and B with the aid of the boundary conditions (8). Then

$$T_n^* = \sqrt{2\pi}\, f_n \frac{D(\xi\rho)}{D(\xi\beta)} \frac{\delta_-(p)}{\alpha - ipu}, \tag{9}$$

where

$$D(xy) = I_n(xy)[xK_n'(x) + hK_n(x)] - K_n(xy)[xI_n'(x) + hI_n(x)].$$

By using the inversion theorem proposed in [8], we obtain inverse Laplace transforms

$$\bar{T}_n = \sqrt{2\pi}\,\delta_-(p) f_n \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{D(\xi\rho)}{D(\xi\beta)} \frac{\exp(\alpha\tau)}{\alpha - ipu}\, d\alpha.$$

The integrand is an analytic function in any finite portion of the plane, with the exception of the points

$$\alpha_0 = ipu \text{ and } \alpha_k = -(\omega_k^2 + p^2),$$

where ω_k are the roots of equation

$$C(\omega_k\beta) = [\omega_k Y_n'(\omega_k) + hY_n(\omega_k)]\, J_n(\omega_k\beta) - Y_n(\omega_k\beta)[\omega_k J_n'(\omega_k) + hJ_n(\omega_k)], \tag{10}$$

$$k = 1,\ 2,\ 3, \ldots$$

Then, according to Cauchy's theorem of residues [8], we have

$$\bar{T}_n = \sqrt{2\pi}\,\delta_-(p) f_n \sum_{k=0}^{\infty} \operatorname{res} \left[\frac{D(\xi\rho)}{D(\xi\beta)} \frac{\exp(\alpha\tau)}{\alpha - ipu} \right]_{\alpha=\alpha_k}.$$

After some necessary calculations, we obtain

$$\bar{T}_n = \sqrt{2\pi}\,\delta_-(p) f_n \left\{ \frac{D(\eta\rho)}{D(\eta\beta)} \exp(ipu\tau) + 2 \sum_{k=1}^{\infty} \frac{C(\omega_k\rho)}{F(\omega_k\beta)} \frac{\omega_k \exp[-(\omega_k^2 + p^2)\tau]}{\omega_k^2 + p^2 + ipu} \right\},$$

$$\eta^2 = p^2 + ipu,$$

$$F(\omega_k\beta) = \omega_k [J_n(\omega_k\beta) Y_n''(\omega_k) - J_n''(\omega_k) Y_n(\omega_k\beta)] + \omega_k\beta [J_n'(\omega_k\beta) Y_n'(\omega_k) - J_n'(\omega_k) Y_n'(\omega_k\beta)]$$

$$+ (h+1)[J_n'(\omega_k) Y_n(\omega_k\beta) - J_n(\omega_k\beta) Y_n'(\omega_k)] - h[J_n(\omega_k) Y_n'(\omega_k\beta) - J_n'(\omega_k\beta) Y_n(\omega_k)]. \tag{11}$$

The inverse transform of function $T_n(\rho, \zeta, \tau)$ is obtained by inversion of the Fourier transforms

$$T_n(\rho, \zeta, \tau) = f_n \int_{-\infty}^{\infty} \left(\frac{1}{2}\delta(p) + \frac{1}{2\pi i p} \right) \left\{ \frac{D(\eta\rho)}{D(\eta\beta)} \exp(ipu\tau) + 2 \sum_{k=1}^{\infty} \frac{C(\omega_k\rho)}{F(\omega_k\beta)} \frac{\omega_k \exp[-(\omega_k^2 + p^2)\tau]}{\omega_k^2 + p^2 + ipu} \right\} \exp(ip\zeta)\, dp.$$

Making use of the well-known equality [7]

$$\int_{-\infty}^{\infty} \varphi(x)\, \delta(x)\, dx = \varphi(0),$$

we obtain

$$T_n(\rho, \zeta, \tau) = \frac{1}{2} E f_n + f_n \sum_{k=1}^{\infty} \frac{C(\omega_k\rho)}{\omega_k F(\omega_k\beta)} \exp(-\omega_k^2\tau) + T_n' + T_n'',$$

where

$$T_n' = \frac{1}{2\pi i} f_n \int_{-\infty}^{\infty} \frac{D(\eta\rho)}{D(\eta\beta)} \frac{\exp[-ip(\zeta - u\tau)]}{p}\, dp; \tag{12}$$

$$T_n'' = \frac{2}{2\pi i} f_n \sum_{k=0}^{\infty} \frac{C(\omega_k\rho)}{F(\omega_k\beta)} \omega_k \int_{-\infty}^{\infty} \frac{\exp[-(\omega_k^2 + p^2)\tau - ip\zeta]}{p(\omega_k^2 + p^2 + ipu)}\, dp; \tag{13}$$

$$E = \frac{\rho^n \left(\frac{h}{n+1} - 1 \right) - \frac{1}{\rho^n} \left(\frac{h}{n+1} + 1 \right)}{\beta^n \left(\frac{h}{n+1} - 1 \right) - \frac{1}{\beta^n} \left(\frac{h}{n+1} + 1 \right)}.$$

The integrand in (12) is an analytic function everywhere in any finite portion of the plane, with the exception of the poles

$$p_0 = 0,\ p_k = iq_k \text{ and } p_{-k} = iq_{-k},$$

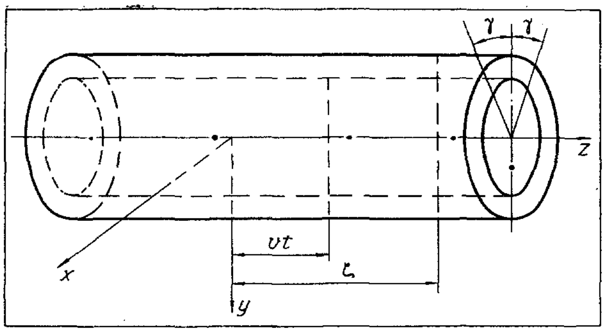

Fig. 1. Schematic drawing of the problem of the temperature field in an unbounded cylinder with a moving heating line.

where

$$q_k=\sqrt{\omega_k^2+\frac{u^2}{4}}-\frac{u}{2}>0;\quad q_{-k}=-\sqrt{\omega_k^2+\frac{u^2}{4}}-\frac{u}{2}<0.$$

In the evaluation of the integral (12), one must consider two cases

$$1)\ \zeta-u\tau>0,\quad 2)\ \zeta-u\tau<0.$$

For function T_n', we obtain, respectively,

$$T_n'=-f_nE-\sum_{k=1}^{\infty}\operatorname{res}\left\{f_n\frac{D(\eta\rho)}{D(\eta\beta)}\frac{\exp[-ip(\zeta-u\tau)]}{p}\right\}_{p=p_k}$$

$$=-f_nE+2\sum_{k=1}^{\infty}f_n\frac{C(\omega_k\rho)}{F(\omega_k\beta)}\frac{\omega_k\exp[q_k(\zeta-u\tau)]}{q_k\sqrt{\omega_k^2+\frac{u^2}{4}}},\qquad(\zeta-u\tau)<0;$$

$$T_n'=-f_nE+2\sum_{k=1}^{\infty}f_n\frac{C(\omega_k\rho)}{F(\omega_k\beta)}\frac{\omega_k\exp[q_{-k}(\zeta-u\tau)]}{q_{-k}\sqrt{\omega_k^2+\frac{u^2}{4}}},\qquad(\zeta-u\tau)>0.$$

In the evaluation of the integral (13), the fraction in the integrand can be expressed in the form of a sum of common fractions [2], then

$$T_n''=f_n\sum_{k=1}^{\infty}\frac{C(\omega_k\rho)}{F(\omega_k\beta)}\omega_k\exp(-\omega_k^2\tau)\left\{-\frac{1}{\omega_k^2+q_k^2}\exp(q_k^2\tau+q_k\zeta)\right.$$

$$\left.\times\operatorname{erfc}\left(q_k\sqrt{\tau}+\frac{\zeta}{2\sqrt{\tau}}\right)+\frac{1}{\omega_k^2+q_{-k}^2}\exp(q_{-k}^2\tau+q_{-k}\zeta)\operatorname{erfc}\left(q_{-k}\sqrt{\tau}-\frac{\zeta}{2\sqrt{\tau}}\right)-\frac{1}{\omega_k^2}\operatorname{erf}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\right\}.$$

The roots ω_k of Eq. (10) can be determined by a method proposed in [9].

NOTATION

φ, r, z	are the cylindrical coordinates;
T_0	is the temperature of a portion of the inner surface of the cylinder;
a	is the radius of outer surface of cylinder;
b	is the radius of inner surface of cylinder;
v	is the speed of heating line;
$\varkappa$	is the coefficient of thermal diffusivity;
t	is the time;
$\delta_-(x)$	is the Heisenberg delta-function;
$\delta(x)$	is the Dirac delta-function;
h	is the heat-transfer coefficient;
I_n, K_n	are the Bessel functions of an imaginary variable with subscript n;
p, α	are the parameters.

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